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LETTER TO THE EDITOR

An approach to the integrability of Hamiltonian systems obtained by reduction

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Abstract. A hierarchy of finite-dimensional integrable Hamiltonian systems can be obtained in a straightforward way by restricting a hierarchy of integrable evolution equations to the invariant subspace of their recursion operator. The independent integrals of motion and Hamiltonian functions for these Hamiltonian systems can be constructed by using the recursion formula and can be shown to be in involution. So these Hamiltonian systems are completely integrable in the sense of Liouville and commute with each other.

It is well known that many finite-dimensional integrable Hamiltonian systems can be obtained by restricting infinite-dimensional integrable Hamiltonian systems to finite-dimensional invariant submanifolds of their phase space (see, for example, [1-6]). We have proposed in [7] a straightforward way to obtain a hierarchy of finite-dimensional integrable systems by restricting the hierarchy of the integrable evolution equations to an invariant subspace of their recursion operator. In this letter, based on our work in [7], we present a method to obtain independent integrals of motion for these systems by using the recursion formula and show them to be in involution. Thus all of these systems are completely integrable Hamiltonian systems in the sense of Liouville [8] and commute with each other.

To illustrate the method, consider the classical Boussinesq hierarchy [9]

$$u_{t_n} = L^n u_x = \begin{pmatrix} Q_{nx} \\ R_{nx} \end{pmatrix} \qquad u = \begin{pmatrix} s \\ r \end{pmatrix} \tag{1}$$

with

$$L = \begin{pmatrix} \frac{1}{2}r & -\frac{1}{4}D^2 + s + \frac{1}{2}s_x D^{-1} \\ 1 & \frac{1}{2}(r + r_x D^{-1}) \end{pmatrix} \qquad D = \frac{\mathrm{d}}{\mathrm{d}x} \qquad D^{-1}D = DD^{-1} = 1.$$

Here no boundary condition for u is required and the integration constant for D^{-1} is defined to be zero. The Lax pair associated with (1) reads

$$\psi_{xx} = M\psi \qquad M = -\zeta^2 + s - \frac{1}{4}r^2 + \zeta r \tag{2}$$

$$\psi_{t_n} = -\frac{1}{2}B_x^{(n)}\psi + B^{(n)}\psi_x \tag{3}$$

where

$$B^{(n)} = \sum_{k=0}^{n} b_k \zeta^{n-k} \qquad b_0 = 1 \qquad b_k = \frac{1}{2}R_{k-1} \tag{4a}$$

$$\begin{pmatrix} Q_{kx} \\ R_{kx} \end{pmatrix} = L\begin{pmatrix} Q_{k-1,x} \\ R_{k-1,x} \end{pmatrix} = L^k \begin{pmatrix} s_x \\ r_x \end{pmatrix}. \tag{4b}$$

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We now consider the following system instead of (2):

$$\psi_{jxx}=M_j\psi_j \qquad M_j=-\zeta_j^2+s-\tfrac{1}{4}r^2+\zeta_j r \qquad j=1,\ldots,N \tag{5}$$

where $\zeta_k\neq\zeta_l$ when $k\neq l$. We call

$$q=(q_1,\ldots,q_N)^{\mathrm{T}}\equiv(\psi_1,\ldots,\psi_N)^{\mathrm{T}}$$
$$p=(p_1,\ldots,p_N)^{\mathrm{T}}\equiv(\psi_{1x},\ldots,\psi_{Nx})^{\mathrm{T}}$$
$$B=\mathrm{diag}(\zeta_1,\ldots,\zeta_N).$$

It was pointed out in [7] that in order to obtain an invariant subspace of L, one has to impose a constraint on potential u as follows:

$$r=\langle q,q\rangle \qquad a=\langle Bq,q\rangle-\tfrac{1}{2}\langle q,q\rangle^2 \tag{6}$$

where $\langle .,.\rangle$ is the inner product on $\mathbb{R}^N$. Under the constraint condition (6), system (5) can be written in canonical Hamiltonian form

$$q_x=\frac{\partial H_0}{\partial p} \qquad p_x=-\frac{\partial H_0}{\partial q} \tag{7}$$

with Hamiltonian function H_0 defined by

$$H_0=\tfrac{1}{2}\langle p,p\rangle+\tfrac{1}{2}\langle B^2q,q\rangle-\tfrac{1}{2}\langle q,q\rangle\langle Bq,q\rangle+\tfrac{1}{8}\langle q,q\rangle^3.$$

By using the recursion operator L, it is shown in [7] that

$$R_k|_A=\sum_{l=0}^{k}h_l\langle B^{k-l}q,q\rangle+2h_{k+1} \tag{8}$$

where the subscript A means to insert (6) into the expression, and h_l are the integrals of the motion for (7). The recursion formula for R_k can be found as

$$R_{k+1}=-\frac{1}{8}\sum_{j=0}^{k-1}\left[R_{jxx}R_{k-j-2}-\frac{1}{2}R_{jx}R_{k-j-2,x}\right]+\frac{1}{4}\left(s-\frac{1}{4}r^2\right)\sum_{j=-1}^{k-1}R_jR_{k-2-j}$$
$$+\frac{1}{4}r\sum_{j=-1}^{k}R_jR_{k-1-j}-\frac{1}{4}\sum_{j=0}^{k}R_jR_{k-j} \qquad k=0,1,\ldots \tag{9}$$

where $R_{-1}=2$, $R_0=r$. Substituting (8) into both sides of (9), a lengthy calculation gives

$$h_{k+2}=\sum_{j=0}^{k-1}\sum_{l=0}^{k-j-1}h_jh_lC_{k+2-j-l}-\frac{1}{2}\sum_{j=1}^{k-1}h_jh_{k+2-j} \qquad k=1,2,\ldots \tag{10}$$

where $h_1=h_2=C_1=C_2=0$, $h_0=C_0=1$,

$$C_{k+3}=-\tfrac{1}{4}\{\langle B^{k+2}q,q\rangle+\langle B^kp,p\rangle+\frac{1}{2}\sum_{m=0}^{k-1}[\langle B^mp,p\rangle\langle B^{k-1-m}q,q\rangle-\langle B^mp,q\rangle\langle B^{k-1-m}p,q\rangle]$$
$$+\tfrac{1}{4}\langle q,q\rangle^2\langle B^kq,q\rangle-\tfrac{1}{2}\langle q,q\rangle\langle B^{k+1}q,q\rangle$$
$$-\tfrac{1}{2}\langle Bq,q\rangle\langle B^kq,q\rangle\} \qquad k=0,1,\ldots.$$

It is clear from (10) that the C_k are also integrals of motion for (7). Indeed it is easy to check by a direct calculation that if (p,q) is a solution of (7), then

$$\frac{\mathrm{d}C_k}{\mathrm{d}x}=0 \qquad \frac{\partial C_k}{\partial q}=-\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial C_k}{\partial p} \qquad k=1,2,\ldots. \tag{11}$$

Define

$$G_k=\sum_{m=0}^{k}[\langle B^mp,p\rangle\langle B^{k-m}q,q\rangle-\langle B^mp,q\rangle\langle B^{k-m}p,q\rangle].$$

It is not difficult to show that the G_k are in involution with respect to the ordinary

Poisson bracket. Then, using the identity

$$\sum_{m=0}^{l} \langle B^{l+k+j-m}p, p\rangle\langle B^m q, q\rangle + \sum_{m=0}^{k} \langle B^m p, p\rangle\langle B^{l+k+j-m}q, q\rangle$$

$$= \sum_{m=0}^{l+k+j} \langle B^{l+k+j-m}p, p\rangle\langle B^m q, q\rangle - \sum_{m=l+1}^{l+j-1} \langle B^{l+k+j-m}p, p\rangle\langle B^m q, q\rangle$$

$$j = 1, 2, \ldots$$

we can show by a straightforward calculation that the C_k are in involution. Since the Vandermonde determinant of $\zeta_1, \ldots, \zeta_N$ is not zero, it is easy to see that grad $C_3, \ldots,$ grad C_{N+2} are independent. So we have the following.

Proposition 1. The Hamiltonian system (7) is completely integrable in the sense of Liouville.

The formula (10) can be rewritten as

$$h_k = C_k + \sum_{l+m+n=k} h_l h_m C_n + 2 \sum_{l+m=k} h_l C_m - \frac{1}{2} \sum_{l+m=k} h_l h_m \qquad k = 1, 2, \ldots \tag{12}$$

where l, m, $n \geqslant 1$, $h_1 = h_2 = C_1 = C_2 = 0$. We find from (12) by induction that

$$h_k = \sum_{l=1}^{k} a_l \sum_{m_1+\ldots+m_l=k} C_{m_1} \ldots C_{m_l} \qquad k = 1, 2, \ldots \tag{13}$$

where $a_1 = 1$, $a_2 = \frac{3}{2}$,

$$a_l = \sum_{i=1}^{l-2} a_i a_{l-1-i} + 2a_{l-1} - \frac{1}{2} \sum_{i=1}^{l-1} a_i a_{l-i} \qquad l = 2, 3, \ldots.$$

We now consider systems stemming from (3)

$$\psi_{jt_n} = -\tfrac{1}{2} B_{jx}^{(n)} \psi_j + B_j^{(n)} \psi_{jx} \qquad B_j^{(n)} = B^{(n)}|_{\zeta=\zeta_i} \qquad j = 1, \ldots, N. \tag{14}$$

Under the constraint condition (6) and (7), (14) becomes by using (8) and (13)

$$\psi_{jt_n} = \frac{1}{2} \sum_{k=1}^{n} \left(\zeta_j^{n-k} p_j \sum_{l=0}^{k-1} h_l \langle B^{k-l-1} q, q\rangle + 2h_k \zeta_j^{n-k} p_j - \zeta_j^{n-k} q_j \sum_{l=0}^{k-1} h_l \langle B^{k-l-1} p, q\rangle \right) + \zeta_j^n p_j$$

$$= \frac{1}{2} \sum_{l=0}^{n-1} h_l \left(\sum_{m=0}^{n-l-1} [\langle B^{n-l-1-m} q, q\rangle \zeta_j^m p_j \right.$$

$$\left. - \langle B^{n-l-1-m} p, q\rangle \zeta_j^m q_j] + 2\zeta_j^{n-l} p_j \right) + h_n p_j$$

$$= -2 \sum_{l=0}^{n-1} h_l \frac{\partial C_{n-l+3}}{\partial p_j} - 2h_n \frac{\partial C_3}{\partial p_j}$$

$$= -2 \frac{\partial C_{n+3}}{\partial p_j} - 2 \sum_{l=1}^{n} \frac{\partial C_{n-l+3}}{\partial p_j} \sum_{i=1}^{l} a_i \sum_{m_1+\ldots+m_i=l} C_{m_1} \ldots C_{m_i}$$

$$= -2 \frac{\partial C_{n+3}}{\partial p_j} - 2 \sum_{i=1}^{n} a_i \sum_{l=1}^{n+3-i} \sum_{m_1+\ldots+m_i=n+3-l} C_{m_1} \ldots C_{m_i} \frac{\partial C_l}{\partial p_j}$$

$$= -2 \frac{\partial C_{n+3}}{\partial p_j} - 2 \sum_{i=1}^{n} a_i \sum_{m_1+\ldots+m_{i+1}=n+3} C_{m_1}, \ldots C_{m_i} \frac{\partial C_{m_{i+1}}}{\partial p_j}$$

$$= \frac{\partial H_n}{\partial p_j} \tag{15}$$

with

$$H_n = -2 \sum_{i=0}^{n} \frac{a_i}{i+1} \sum_{m_1+\ldots+m_{i+1}=n+3} C_{m_1} \ldots C_{m_{i+1}} \qquad (a_0 = 1).$$

Equation (15) and (11) means that under the constraint condition (6) and (7), (14) becomes a Hamiltonian system

$$q_{t_n} = \frac{\partial H_n}{\partial p} \qquad p_{t_n} = -\frac{\partial H_n}{\partial q} \qquad n = 1, 2, \ldots. \tag{16}$$

Proposition 2. The Hamiltonian systems (16) ($n = 0, 1, \ldots, H_0 = -2C_3$, call $t_0 = x$) are completely integrable in the sense of Liouville and commute with each other. If (p, q) satisfies (7) and (16) then u given by (6) solves the evolution equation (1).

Proof. Notice that

$$\frac{\mathrm{d}C_k}{\mathrm{d}t_n} = \{H_n, C_k\} = 0 \qquad \{H_k, H_m\} = 0 \qquad k, m = 0, 1, \ldots.$$

The $C_3, \ldots, C_{N+2}$ are also the N independent integrals of motion in involution for (16). Thus (16) ($n = 0, 1, \ldots$) are completely integrable Hamiltonian systems and commute with each other. Since (1) is deduced from the compatibility condition of (5) and (14), (7) and (16) are obtained by substituting (6) into (5) and (14), respectively, it follows that if (p, q) satisfies (7) and (16) then u given by (6) solves (1).

The approach proposed above is general. It can be used to other infinite-dimensional Hamiltonian systems. For example, for the AKNS hierarchy [10]

$$u_{t_n} = 2\mathrm{i}JL^n u \qquad u = \begin{pmatrix} r \\ q \end{pmatrix} \qquad J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{17}$$

where

$$L = \frac{1}{2\mathrm{i}} \begin{pmatrix} D - 2rD^{-1}q & 2rD^{-1}r \\ -2qD^{-1}q & -D + 2qD^{-1}r \end{pmatrix}$$

the associated eigenvalue problem is

$$\psi_x = M\psi \qquad M = \begin{pmatrix} -\mathrm{i}\zeta & q \\ r & \mathrm{i}\zeta \end{pmatrix} \qquad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{18}$$

and the time evolution equation of ψ is

$$\psi_{t_n} = N^{(n)}\psi \qquad N^{(n)} = \begin{pmatrix} A_n & B_n \\ C_n & -A_n \end{pmatrix} \tag{19}$$

where

$$A_n = \sum_{k=0}^{n} a_k \zeta^{n-k} \qquad B_n = \sum_{k=1}^{n} b_k \zeta^{n-k} \qquad C_n = \sum_{k=1}^{n} c_k \zeta^{n-k}$$

$$\begin{pmatrix} c_k \\ b_k \end{pmatrix} = L^{k-1}u \qquad a_0 = -\mathrm{i} \qquad a_k = D^{-1}(qc_k - rb_k) \qquad k = 1, 2, \ldots, n.$$

Now consider the system

$$\psi_{jx} = M_j\psi_j \qquad \psi_j = \begin{pmatrix}\psi_{1j}\\ \psi_{2j}\end{pmatrix} \qquad M_j = \begin{pmatrix}-\mathrm{i}\zeta_j & q\\ r & \mathrm{i}\zeta_j\end{pmatrix} \qquad j = 1, \ldots, N \tag{20}$$

where $\zeta_k \neq \zeta_l$ when $k \neq l$. We call

$$\Phi_1 = (\psi_{11}, \ldots, \psi_{1N})^{\mathrm{T}} \qquad \Phi_2 = (\psi_{21}, \ldots, \psi_{2N})^{\mathrm{T}} \qquad B = \mathrm{diag}(\mathrm{i}\zeta_1, \ldots, \mathrm{i}\zeta_N).$$

To obtain an invariant subspace of L, we get a constraint on u [7]

$$r = \langle\Phi_2, \Phi_2\rangle \qquad q = -\langle\Phi_1, \Phi_1\rangle. \tag{21}$$

Under the constraint condition (21), (20) can be written in canonical Hamiltonian form

$$\psi_{1jx} = \frac{\partial H_0}{\partial\psi_{2j}} \qquad \psi_{2jx} = -\frac{\partial H_0}{\partial\psi_{1j}} \qquad j = 1, \ldots, N \tag{22}$$

with $H_0 = -\frac{1}{2}\langle\Phi_1, \Phi_1\rangle\langle\Phi_2, \Phi_2\rangle - \langle B\Phi_1, \Phi_2\rangle$.

It was shown in [7] by using recursion operator that

$$\begin{aligned} c_k|_A &= \sum_{l=0}^{k-1} h_l\langle B^{k-1-l}\Phi_2, \Phi_2\rangle(-\mathrm{i})^{k-1-l} \\ b_k|_A &= -\sum_{l=0}^{k-1} h_l\langle B^{k-1-l}\Phi_1, \Phi_1\rangle(-\mathrm{i})^{k-1-l} \end{aligned} \qquad h_0 = 1 \tag{23}$$

where h_l are the integrals of motion for (22). Using (23) and the recursion formula for a_k, b_k and c_k

$$c_{k+1} = -\frac{\mathrm{i}}{2}c_{kx} + \frac{1}{2r}\sum_{l=1}^{k-1}\left(\tfrac{1}{4}c_{lx}c_{k-l,x} - \mathrm{i}c_{lx}c_{k-l+1} - c_{l+1}c_{k-l+1} + r^2c_lb_{k-l}\right)$$

a straightforward computation yields

$$h_k = \sum_{l=0}^{k-1}\sum_{m=0}^{k-l-1} h_mh_lF_{k-m-l} - \frac{1}{2}\sum_{l=1}^{k-1} h_lh_{k-l} \tag{24}$$

where

$$F_1 = -\mathrm{i}\langle\Phi_1, \Phi_2\rangle$$

$$\begin{aligned} F_k = (-\mathrm{i})^k\langle B^{k-1}\Phi_1, \Phi_2\rangle &+ \frac{1}{2}\sum_{j=0}^{k-2}(-\mathrm{i})^k[\langle B^j\Phi_1, \Phi_1\rangle\langle B^{k-2-j}\Phi_2, \Phi_2\rangle \\ &- \langle B^j\Phi_1, \Phi_2\rangle\langle B^{k-2-j}\Phi_1, \Phi_2\rangle] \qquad k = 1, 2, \ldots. \end{aligned}$$

Thus the F_k are integrals of motion for system (22). In a similar way, we can show that the F_k are in involution with respect to the ordinary Poisson bracket and that grad $F_1, \ldots,$ grad F_N are independent. This implies that the Hamiltonian system (22) is completely integrable. Similarly, we get

$$h_k = \sum_{l=1}^{k}\bar{a}_l\sum_{m_1+\ldots+m_l=k} F_{m_1}\ldots F_{m_l} \tag{25}$$

where $m_1 \geqslant 1, \ldots, m_l \geqslant 1$, $\bar{a}_1 = 1$, $\bar{a}_2 = \frac{3}{2}$,

$$\bar{a}_l = 2\bar{a}_{l-1} + \sum_{m=1}^{l-2}\bar{a}_m\bar{a}_{l-1-m} - \frac{1}{2}\sum_{m=1}^{l-1}\bar{a}_m\bar{a}_{l-m} \qquad l = 2, 3, \ldots.$$

For the system obtained from (19)

$$\psi_{jt_n} = N_j^{(n)}\psi_j \qquad N_j^{(n)} = N^{(n)}|_{\zeta=\zeta_j} \qquad j=1,2,\ldots,N. \tag{26}$$

It can be shown in the same way that under the constraint conditions (21) and (22), (26) can be written in canonical Hamiltonian form

$$\psi_{1jt_n} = \frac{\partial H_n}{\partial \psi_{2j}} \qquad \psi_{2jt_n} = -\frac{\partial H_n}{\partial \psi_{1j}} \qquad j=1,2,\ldots,N \tag{27}$$

where

$$H_n = \sum_{l=0}^{n} \frac{\bar{a}_l}{l+1} \sum_{m_1+\ldots+m_{l+1}=n+1} F_{m_1}\ldots F_{m_{l+1}} \qquad (\bar{a}_0=1) \qquad n=1,2,\ldots.$$

Finally we have the following.

Proposition 3. The systems (27) ($n=0,1,\ldots$, call $t_0=x$) are completely integrable and commute with each other. If Φ_1 *and* Φ_2 satisfy both (22) and (27) ($n=1,2,\ldots$), then u given by (21) solves (17).

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